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THE TEACHING OF PRIMARY ARITHMETIC

A critical study of recent tendencies in method

BY

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INTRODUCTION

BY DAVID EUGENE SMITH

The evolution of the teaching of primary arithmetic extends over a period of about two hundred years, although numerous sporadic efforts at teaching the science of number to young children had been made long before the founding of the Francke Institute at Halle. During the eighteenth century not much progress was made until there was established the Philanthropin at Dessau, and perhaps it would be more just to speak of primary arithmetic as having its real beginning in this institution at about the time that our country was establishing its independent existence. It is, however, to Pestalozzi, at the beginning of the nineteenth century, that we usually and rightly assign the first sympathetic movement in this direction, and it is the period from that time to the present that has seen the real evolution of the teaching of arithmetic to children in the first school years.

INTRODUCTION

The evolution of this phase of education is one of the most interesting and profitable studies that a teacher of arithmetic can undertake. It reveals the experiments, many of them puerile but a few of them virile, that have been made from time to time; it brings into light the failures which should serve as warnings, and the successes which will inspire the teacher to better and more carefully considered effort; it shows the trend of primary instruction, and it makes the student of education more sympathetic with the great problem before him. In particular, he will see the danger of narrowness in matters of method, the futility of expecting to create genuine interest by any single line of devices, the pitiful results that came from over-emphasis of the doctrine of formal discipline, and the errors of judgment that have been made in deciding upon what constitutes reality in an arithmetical problem for children. If by such a study he should see the childishness of confining one's self to the use of cubes alone, or of some special type of number chart, or of some particular form of number cards, or of sticks of varied lengths or

shapes, the result would be salutary. If by his study of the absurd extreme to which formal discipline was carried a generation ago he is led to see the equal absurdity to which so many teachers are tending to-day — the denying that any such discipline exists at all — his labor will bear good fruit in the school room of the present.

It fortunately happens that after this century of experiment we are getting about ready to take some account of stock; to weigh up values; to select from what the world has produced, and to select with some approach to good judgment. It is not probable that the time is entirely ripe for this labor, because we are at present in the midst of a period of agitation that seems certain to warp our judgment, the period of agitation for a somewhat narrow phase of industrial education; but even with this danger we are better able to weigh up the values in the teaching of arithmetic than we have ever been before. One reason for this is that we now have men and women of sufficiently sound education and sufficiently broad view to attack the problem. These men appreciate the efforts of Pestalozzi, but they

recognize that these are to the present what the science of Franklin is to that of Kelvin and Thompson. They appreciate what Tillich did for number work, and the influence of Grube; but they know that these men were narrow in view and dogmatic in statement, and that they stand rather as warnings than as founders of any worthy theory.

In looking for such a man to prepare a report upon the teaching of arithmetic in the primary grades, the American members of the International Commission on the Teaching of Mathematics turned first of all to Professor Suzzallo.[1] They felt that his standing as a scholar, his experience as a practical school man, and his position in the educational world fitted him perfectly for a work of this importance. That their good judgment did not fail them will be seen in reading the following report. In it Professor Suzzallo

[1] The material presented in this study was originally collected and organized for the purposes of a special report to a sub-committee of the International Commission on the Teaching of Mathematics. This commission was created at the International Congress of Mathematicians held in Rome in 1908. The report was first published in the *Teachers College Record*, March, 1911.

has set forth very clearly the aims of instruction in primary arithmetic, rightly considering these aims in their evolutionary rather than in their static aspects, emphasizing the importance of this phase of the study, and showing the present tendencies that seem making for a more rational view of teaching. He has discarded the narrow and trivial concept of method that characterized the educational work of a generation now passing away, and has brought dignity to the term by considering it from the modern scientific standpoint. He has discussed both historically and psychologically the important question of object teaching, showing the failures that have resulted from narrow views of the purpose of such an aid and the success that may be expected from a more rational use of number material. The whole question of the rôle of reason, or rather of the pupil's effort to express the rationalizing process has been considered, the extreme danger points have been indicated, and the bearing of some of our saner forms of psychology upon the subject has been set forth. The technique of number, including the question of accuracy and speed

in the operations, has also been treated in a very acceptable manner, Professor Suzzallo's experience in this phase of work having been unusual. And finally, the vexed question of what constitutes a genuinely concrete problem has been considered in its various bearings, and the present tendencies in problem-making have been indicated. It is needless to say that the last word has not been spoken on this phase of the work, and that it never will be. New generations produce new lines of application of arithmetic, even for children. But with respect to the general principles of the selection of matter for the framing of problems, Professor Suzzallo speaks with a conviction that will carry weight.

With all of the opinions expressed in such a report probably no reader will agree. It would be a poor discussion that would not provoke some opposition. But with the general tenor of the report it is certain that most thinking teachers of to-day will find themselves in hearty accord. More important still is the fact that the report sets forth in clear language the present status of primary arithmetic in the more thoughtful edu-

cational circles in this country, and that it will state to teachers at home and abroad the tendencies as they now appear to the leaders of educational thought in the United States.

Teachers College, Columbia University,
February, 1911.

I

THE SCOPE OF THE STUDY

It is the function of this study to convey some notion of the methods employed in teaching mathematics in the first six grades of the American elementary school. No attempt is made to give a minute description of the endless details of teaching procedure, nor even to enumerate all the types of teaching method employed. Its purpose is restricted to an analysis of the larger tendencies in teaching practice which are representative of the spirit of mathematical instruction in the lower schools.

Function of the Study to Trace General Tendencies

We should have a much simpler task if it were ours to sketch the purposes of mathematical instruction, or to outline the nature and organization of the various mathematical courses of study. As it is, we have to describe something less concrete, namely the method employed in the pre-

sentation of mathematical subjects. Intimately dependent upon the subject matter involved, dominated by the special aims of mathematical instruction in a non-technical school, adjusted to the immaturities of childhood, and reflecting the personal habits of mind of the teacher, — teaching method emerges — a powerful, variable, and subtle thing. In spite of subtlety and variability there are, however, certain general practices that can be described and analyzed. It is with these that this study will deal.

Teaching Method is a Mode of Presentation

Owing to the existing confusions, it is well at the very outset to have in mind a clear definition of the term "teaching methods." Teaching methods are always methods of presentation. In this respect the teaching art is like any other art, literary, graphic, plastic, or what not. The literary artist, for example, has a purpose, a subject matter, a particular audience, and a special style of presentation. All these factors are present in the teaching art. The aims of instruction, the particular facts to be taught, the immaturity

of the child taught, and the inevitable personality of the teacher determine the style of instruction, or, to use our own "trade word," a method of teaching. Every teacher, then, has a style or method — conscious or unconscious, good, bad, or indifferent. Unlike the literary artist, he has many ends to serve rather than one. His functions are general to life, and include moral, social, and personal ends, as well as those that are æsthetic. His methods of communication, too, are more than one. He presents his experiences objectively and graphically, as well as through the medium of written words and speech. Always the teacher's end is to stimulate growth through the presentation of experiences. When that presentation takes a form and order different from that usual to adult life for the precise purpose of making the fact more readily comprehensible to the immature mind of the child, then that modification may be called a method of teaching. Teaching methods are always special manners of readjusting adult wisdom to the special psychological conditions of a student's mind.

Distinct Uniformities Exist among its Variations

In the concrete, methods of teaching are always specialized responses to situations, and as variable as situations are variable. Life is never just the same at any point. Yet certain essential similarities exist and give us the opportunity to interpret life in terms of law. The same may be said of the teaching life. In a sense it never repeats itself; yet to the degree that the same end, the same subject matter, and the same immaturity of mind recur in class rooms, teachers will tend to use similar modes of adjustment. In describing mathematical teaching in the primary schools, it is these similar modes of teaching adjustment, these similar "general methods," that we shall describe and analyze.

The Methods of Public Elementary Schools are Representative

It will be unnecessary to have a separate treatment of the "general methods" of mathematical teaching for public schools and private schools. Whatever may be said of the state-sup-

ported schools will in general be true of private institutions. It is true of elementary schools as it is not of secondary and higher schools, that private institutions hold a relatively minor place, as compared with public or state schools. They are in a sense mere adjuncts to the public school system, claiming, in the generality of cases, no real difference in their ideals and methods of instruction. In the larger number of cases they draw their courses of study and methods from the public school systems of the immediate neighborhood. The social status of the parent or the personal incapacity of the child, rather than difference of school methods, is the cause of the special clientage of the private primary schools in the United States. Hence, a description of the characteristic methods of the public schools will be representative of the prevailing modes of instruction in American private elementary schools.

Elementary Mathematics is Mainly Arithmetic

Mathematical instruction in the first six years of the elementary school concerns itself almost exclusively with the teaching of arithmetic. A

decade or so ago there was a vigorous movement for the introduction of algebra and geometry into the elementary school. As a result, these subjects made their appearance in the seventh and eighth grades — seldom in the first six school years. Where the influence of this movement penetrated the lower grades or persisted in the higher grades, the algebraic and geometric elements involved were so restricted and simplified that they became part and parcel of the subject of arithmetic, rather than the elementary phases of two more advanced subjects. This is true of the simple algebraic equation or the measurement of simple geometric figures where introduced below the seventh year.

Elementary Arithmetic Emphasizes the Four Fundamental Processes

By common practice, even the arithmetic taught in the primary grades has been given a restrictive emphasis. For the most part it is concerned with the mastery of the fundamental processes in manipulating integers and fractions. The casual observer, in reading American courses

of study, will note that in the lower grades the mathematical subjects taught are named after the abstract process involved rather than after the particular business institution to which the arithmetic is concretely applied. Thus in the lower grades we teach counting, subtraction, fractions, decimals, percentage, etc.; while in the higher grades we teach interest, stocks and bonds, commission, insurance, etc. There is of course no hard and fast line of demarcation; one emphasis gradually passes over into the other, an approximate balance being maintained in the intermediate grades. It will perhaps simplify the task of this study and make its treatment more thoroughly representative of all conditions, if the general methods described be restricted to that field which is most characteristic of the first five or six years of mathematical instruction, namely, to the teaching of the fundamental processes of manipulating integers and fractions along with their simple applications to concrete problems. This discussion, then, will be limited to the period of school life in which the tools of arithmetic are acquired.

The Need for Studying Exceptional Reform Tendencies

While the aspects of mathematical instruction here studied and presented are selected because of their representative nature, it would be unwise to restrict ourselves to a statement of the commonly accepted procedures of school-room practice. There are in America certain reform tendencies which are as characteristic of conditions as are the conservative practices. These modifying forces need to be mentioned along with the practices that they alter. Again, there are certain scientific effects now well under way to study the problem of methods in teaching. While these have, as their immediate aim, the acquisition of new knowledge rather than direct educational reform, their ultimate effect will be to change methods of teaching. For this reason they are important, and have a proper place in this presentation.

II

THE INFLUENCE OF AIMS ON TEACHING

Factors Influencing Teaching Methods

It has been suggested that all teaching methods represent adjustments to several variable factors in the school-room situation. Teaching method is never, or should not be, just one thing. It is as variable as the factors that determine its situation. The purposes of mathematical instruction, the nature of the fact to be taught, the immaturity of the child, the teacher's scholarly equipment, his personality, his attitude toward the very idea or institution of method,—all these are influences in determining the status of mathematical teaching. Some of them are so important that it will be necessary to discuss them in detail at the very outset.

The Influence of a Scientific Aim

The purposes of mathematical instruction in the elementary school must always be very in-

fluential upon method. It makes a great difference whether one is merely teaching the elements of mathematics or is teaching mathematics as a tool for business life. It has not been long since the aim of mathematical teaching was merely scientific. The facts taught were the beginning of a science, and the end was to obtain a foundation for more advanced facts of the same kind, which were dependent upon this foundation. As the teacher had learned his mathematics, so he taught the subject. To a considerable degree, as the master's adult mind classified the facts of the subject, so he presented it to the child. His methods were logical rather than psychological. He gave the finished product or process to the child without special adaptation to the child's immaturity; a roundabout method that slowly approximated and only finally achieved the full result was with such a teacher exceptional.

Such a scientific aim, implicit rather than expressed, dominated the methods of teaching when arithmetic was handed over to the elementary schools by the higher institutions of education. The first purpose to be rooted in the traditions

of mathematical teaching, it still persists with all the rigidity of a conservative force. Teachers still tend to teach future workmen in the lower schools as they themselves were taught by scientific scholars in the universities. And high school and college instructors still impose their standards upon the lower schools so as to influence their methods of instruction. As higher instruction still remains largely scientific in purpose and method, its effects reinforce the earliest tradition in the elementary schools. Under such an influence, the worth of a mathematical fact is measured by its place in a logical scheme, rather than by its significance and recurrence in everyday life. The mathematician may need to know all about the names of the places in notation and numeration; the layman cares only about the accurate reading and writing of numbers, and not at all about the verbal title of "units of thousands" place. Again the rational needs of a student of mathematics may require an understanding of the reasons why we "carry" in column addition, but the effective everyday use demands an accurate *habit* of "carrying" rather than an

accurate *explanation*. Yet just such methods persist in our schools because of the domination of a scientific treatment of the subject.

The Influence of the Aim of Formal Discipline

The remoteness of such mathematical teaching from the needs of common life constantly threatens the loyalty and support of the public. Some defense becomes necessary on other than scientific grounds. Such a sanction could not be found in utilitarianism, for the waste was evident. It remained for a psychological theory to sketch a defense upon "disciplinary" grounds. The doctrine of "formal discipline" says that such mathematical teaching trains the powers of the mind so that any mastery gained in mathematics is a mastery operating in full elsewhere, regardless of the remoteness of the new situations from those in connection with which the power or ability was originally acquired. The facts and processes mastered may not be those most needed in daily life, but they are good for every man inasmuch as they train his mind. Such was the dictum of the doctrine of "formal discipline."

The effect of such doctrine is to defend and perpetuate every obsolete, unimportant, and wasteful practice in the teaching of mathematics. No matter that partnership as taught in the schools had its original sanction in its close correspondence to the reality of business practice; no matter that the old sanction has passed; teach it now for its ability to discipline the mind! No matter that "life insurance" touches more men than "cube root"; the latter should be kept because of its power to train the mind. In life, where "approximation" of amounts suffices, the teacher demands absolute accuracy, and the ethical worth of such precise truth is the high law for its defense. In life, "the butcher, the baker, and the candlestick maker" figure out the total of a bill mainly "in their heads," with a few accessory pencil scribbles upon paper; the teacher finds sanction in æsthetics for requiring a complete statement written or re-written in exquisite form. Regardless of the truth that is concealed in the doctrine of "formal discipline," it must be confessed by those who know the history of teaching method in the United States that it is

the main defense of conservatism and the largest cause of waste in teaching methods.

The Shift in Emphasis from Academic to Social Aims

Such has been the ground upon which recent educational reform has operated. Slowly the older scientific and disciplinary aims of instruction have given way to the newer purposes of business utility and social insight. In that step a large transition has been covered. Before, the school measured the worth of its work by standards internal to educational institutions. The schoolmaster and the scholar, rather than the man on the street, had formulated the scientific classifications of mathematics and expounded the doctrine of "formal discipline." Thereafter, the measure of efficient school instruction was determined by standards external to the school, the product of conditions outside of school life. Business need and social situation determine whether a fact or a process is worth comprehending, and whether the method of instruction has been effective.

Business Utility as an End

The utilitarianism that first attacked the older course of study and its methods was the utility of the business world. The arithmetic of business life became the standard. The practices of the market determined what matter, skill, and accuracy should be demanded of the elementary school pupil. Recently it became the habit to call upon the business man to give his opinion as to what constitutes good arithmetical training; and no criticism was so feared as that of the business leader who said that the boys that came to him were incompetent. Committees on courses of study have even investigated the relative frequency and importance of specific arithmetical processes in the business world with the idea of utilizing the results as a basis for changes in the mathematical curriculum.

This aim of business utility, coming at a time when the elementary school course was felt to be overcrowded, met with a ready reception. It operated for the time being as the standard by which materials and methods in arithmetic were

to be eliminated, if not actually selected. Materials not general to the business world, such as the table of Troy weight, were therefore eliminated. Processes of computing interest infrequently used were supplemented by more widespread and up-to-date methods. More doing and less explaining characterized the instruction in adding columns of figures, and such manipulation mimicked the exact conditions of its use in the world at large. If strings of figures are usually added in vertical columns in the business world, then they should be taught in vertical columns more nearly exclusively than before. The obsolete and the relatively infrequent, the over-complex and the wasteful processes of the old arithmetic tended to disappear. More than any other influence, this aim of business utility has combated the over-conservative influence of scientific and disciplinary aims which dominated previous decades. The newer methods of teaching have kept the best of the old movements. The work is still scientific in that it is accurate; it is still disciplinary in that it trains; but the truth and the training which are given are

selected by and associated with actual business situations common to every-day life.

Broad Social Utilitarianism as a Standard

There is evidence in the present thought of teachers that a utility broader than that of the business world is beginning to obtain in the schools. Everywhere in these days the American teacher and the educational writer speak of the social aims of education. More than ever before, the social consciousness of the teacher enlarges. This general increase in the social consciousness of the teacher is reflected in mathematical instruction.

In spite of an increasing movement toward specific vocational training, — as seen in the industrial education movement, for example, — there has been a reactionary defense of the elementary schools as an institution for very general training in the things that are socially fundamental and common. This movement toward the preservation of the elementary school as a place for giving a broadly socialized and modern culture not only is checking the inroads of a

narrow vocational education, but is broadening the conception of the older studies, of which arithmetic is but one. Arithmetic is not a subject in which only the skills of calculation are cultivated; it is one that contributes social insight, just as history and geography do.

The influence of the social aim of instruction upon mathematical instruction is subtle but obvious. The business man's opinion with reference to arithmetical instruction is not always taken as gospel. There are other standards. "Why," says the schoolmaster, "should I train people for your special needs, any more than for the demands of other trades that men ply? To be sure, our graduates do not fit perfectly into your shop at once. But that precise and local adjustment is the work of the business course or of shop apprenticeship. My function is to train men for the situations common to all men and special to no class. The elementary school is a school for general culture or social appreciation, not a business college or a trade school." The sociologist usurps the place of the business man as the school's proper critic.

Some Concrete Effects of the Change in Aim

The immediate effect is that arithmetical applications find a larger place in teaching. A saner relation is established between abstract examples and concrete problems. And the problems, in increasing extent, are real problems, typical of life, if not actual. No more does arithmetic, in the best schools, confine itself to figures alone. Figures are applied in concrete problems. There may be days of teaching when not a figure is used during the arithmetic period. The social setting, the business institution, which calls for the calculation, is studied as carefully as the process of calculation. The students are given a knowledge of banking as well as skill in the computation of interest. They may even visit a bank, a factory, a shop, as the case may require. Instead of having fifteen problems that deal with fifteen different subjects all more or less remote from one another, as was almost universally the case with older text-books and teaching methods, the class hour may be given over to fifteen problems related to one situation, such

as might develop in the business of a bakery shop or an apartment house. Thus arithmetic gradually gains social setting and unity.

To-day teaching methods in arithmetic are in a state of transition : old and new purposes mingle with unequal force and give us a mixed process of instructing. Old materials and methods still persist, for logical and disciplinary ideals still hold ; but the newer regimen ushered in by the demands of business utility and social understanding gains ground. The obsolete, the untrue, the wasteful methods pass from arithmetic teaching ; and the pressing, modern, and useful activities and understandings enter. Arithmetic is less abstract and formal as a subject than it was ; it has become increasingly vital and concrete with real interests, insights, and situations. The grind of sheer mechanical drill decreases in teaching ; and a reasoned understanding of relations, in some degree at least, is substituted. Artificial motives and incentives are less frequently used to get work done, while the quantitative needs of the child's life and the intrinsic interest of children in the institutional occupations of their elders provide a more vital motive for the use of arithmetic.

III

THE EFFECT OF THE CHANGING STATUS OF TEACHING METHOD

Method as a Psychological Adjustment to the Child

At the very beginning, it was suggested that many factors enter into the nature of our teaching methods. There was occasion to show the effect of varying aims on the spirit and manner of instruction, for the end in view inevitably influences any presentation of facts, in school or out. The most significant factor, however, in teaching method is the attempt to adjust methods of presentation to the psychological conditions of childhood. Teaching method in the school is primarily a mode of presentation designed to stimulate the energies of children. As long as the teacher was the most active person in the classroom, method as such was not important in pedagogical theory. The focusing of attention on the child as an active human factor to

be given careful consideration is responsible for the extended development of teaching technique. The growing importance of "method" in educational theory marks a growth in the teacher's consciousness of psychological factors, precisely as the appearance of the newer aims in teaching has marked an increased regard for social factors.

The Effect of an Increased Reverence for Childhood

Two important movements have been responsible for the development of a psychological consciousness of the pupil as a dominating factor in teaching method. One is humanitarian; the other, scientific.

There has been a steady growth in reverence and sympathy for childhood. As yet it has scarcely expressed itself with fullness. Its presence is revealed by the widespread enactment of laws designed to guarantee the rights of childhood — laws against child labor and in favor of compulsory education. The growth of special courts for juvenile offenders, the development of playgrounds, and the decreased brutality of dis-

cipline at home and school, are other symptoms of the public attitude toward childhood. The wide acceptance of the "doctrine of interest" in teaching; the enrichment of the curriculum; specialized schools for truants and defectives; individual instruction, — these are the schoolmaster's recognition of the modern attitude toward childhood. Under such conditions teaching becomes less and less a ruthless external imposition of adult views, and more a means of sympathetic ministry to those inner needs of child life which make for desirable qualities of character. While it is true that teaching method is a condescension to childhood, it is a socially profitable condescension in that it is a guarantee of more effective and enduring mastery of the life that is revealed at school. Since the child's acquisition tends the more to be part and parcel of his own life under such sympathetic teaching, the products of such instruction are enduring.

The Reconstruction of Method through Psychology

Such a humanitarian movement naturally called for knowledge of the child — the wisdom

of common sense soon exhausts itself, and more scientific data are demanded. Thus the "child study movement" came into existence. The movement was in some degree disappointing, for frequently it busied itself in cataloguing the obvious rather than in classifying new and hitherto unexplained data. But one thing it did: it focused attention upon the child as the crucial factor in education, the prime conditioning force in all methods of instruction. Since then, a saner psychological foundation has been laid for educational procedure, one which is criticising and reconstructing teaching method at every turn. Hitherto, teaching methods had been improved fitfully through a crude empiricism. As the ablest teachers became dissatisfied with their teaching and dared to vary their methods, they selected the successful experiments, and other teachers willingly adopted the methods that seemed better than their own. Now a body of general psychological knowledge, rich in its criticism of old methods and in its suggestion of new means of procedure, gives a scientific basis to teaching method. Where additional psy-

chological knowledge is needed, the educational psychologists seek it through special investigations. And where the counter claims of competing methods defy ordinary psychological analysis and investigation, judgment is sought through an experimental pedagogy which submits teaching processes to comparative tests under normal classroom conditions.

The Increased Professional Respectability of Conscious Method

Increased sympathy with childhood and increased scientific knowledge of human nature together give teaching method a new justification. The result is that the era of complete dependence upon teaching genius and mere common sense in methods of instruction has passed out of the American elementary school. We are now in a period where a specific professional technique in teaching is demanded, a technique partly developed out of crude personal and professional experience, and partly founded upon scientific criticism and experiment. A new humanitarian and scientific attitude toward the mental life of

children elevates teaching method to a position it has never before enjoyed.

The public elementary-school teacher is conservative indeed who will deny that there is anything worthy in the notion of "method." As a class, teachers have faith in the special professional technique which is included under the term. They are critical of the many abuses which have been committed in the name of method. Method cannot be a substitute for scholarship. It cannot be a "cut and dried" procedure indiscriminately or uniformly applied to class-room instruction. Like every other technical means, teaching method is subject to its own limitations and strengths, a fact which the average teacher recognizes.

The Prevalence of Methods Emphasizing a Single Idea

In spite of the fact that the majority of elementary teachers keep reasonably sane on the problem of method in teaching, it must be admitted that a considerable proportion of teachers are inclined to be attracted by systems of method

that greatly over-emphasize a single element of procedure. The hold which the "Grube method," with its unnatural logical thoroughness and progression, gained in this country two or three decades ago is scarcely explicable to-day. Scarcely less baffling is the very large appeal made by a series of textbooks which lays the stress upon the acquisition of arithmetic through the idea of ratio and by means of measuring. Manual work as the source of arithmetical experiences is another special emphasis, which, like the others, has had its enthusiastic adherents. Again it is "arithmetic without a pencil" or some other over-extension of a legitimate local method into a "panacea" or "cure-all," which confronts us. The promulgation and acceptance of such unversatile and one-sided systems of teaching method are indicative of two defects in the professional equipment of teachers: (1) the lack of a clear, scientific notion as to the nature and function of teaching method, and (2) the lack of psychological insight into the varied nature of class-room situations. Untrained teachers we still have among us, and others, too, to whom a little

knowledge is a dangerous thing. These are frequently carried away by the enthusiastic appeals of the reformer with a system far too simple to meet the complex needs of human nature. Our experiences seem to have sobered us somewhat, the increase of supervision has made responsible officers cautious, and increased professional intelligence has put a wholesome damper upon naïve and futile proposals to make teaching easy.

The Tendency toward Over-Uniformity in Method

A more serious evil than that just mentioned is the tendency of the supervising staff to over-prescribe specific methods for class-room teachers. Recently there has developed, more particularly in large city systems, a tendency to demand a uniform mode of teaching the same school subject throughout the city. The prime causes of this tendency are to be found (1) in the specialization of grade teaching, and the consequent interdependence of one teacher on another; and (2) in the mobility of the school population,

which involves considerable lost energy if teachers do not operate along similar lines.

The result of such imposed uniformity is a reduction of spontaneity in teaching. The process of instruction proceeds in a more or less mechanical fashion, the teacher working for bulk results by a persistent and general application of the methods laid down. That teaching which at every moment tends to adjust itself skillfully to the changes of human doubt and interest, difficulty and success, discouragement and insight, now taking care of a whole group at once, now aiding an individual straggler, now resolutely following a prescribed lead, now pursuing a line of least resistance previously unsuspected, cannot thrive under such conditions. The demand for an excessive uniformity stifles teaching as a fine art, and makes of it a mechanical business; only those activities that fit the machine can go on. Thus it happens that we memorize, cram, drill, and review; and soon the subtler processes of thinking and evaluating, which are the best fruit of education, cease to exist.

Method as a Series of Varied, Particular Adjustments

Fortunately the one-method system of teaching will soon belong to the past; and fortunately, too, the imposition of uniform methods is beginning to lose ground, even in our cities. For the most part, the common sense of teachers and the positive statements of our better theorists keep teaching methods in a sane and useful status. Teaching methods should be as infinitely variable as the conditions calling for their use are endlessly changeable. Not one method but many are necessary, for methods are supplementary rather than competitive. No one method should be used with a pre-established rigidity; each must be flexible in its uses, so as to accomplish the varied work to be done. The teacher, with his everyday contact with the problems of childhood, is the best interpreter of conditions and the best chooser of the tools of instruction. The supervisor may criticize, suggest, and advise; he may call attention to fundamental principles involved; but the teacher himself must

finally choose his own methods. He is the only one who can know conditions well enough to adjust teaching methods to the needs of his own children.

Arithmetic teaching has suffered from false uses of teaching method. In this respect it has shared the common professional lot. But in addition it has had special difficulties and adventures of its own. We have now to note those special phases of teaching method which are peculiar and local to mathematical instruction.

IV

METHOD AS AFFECTED BY THE DISTRIBUTION AND ARRANGEMENT OF ARITHMETICAL WORK

The Tendency toward Shortening the Time Distribution

SEVERAL decades ago arithmetic, as a formal subject, was begun in the first school year and continued throughout the grades to the last school year. This is no longer a characteristic condition, much less a uniform one. There have been forces operating to complete the subject of arithmetic prior to the eighth year, and to delay its first systematic presentation in the primary grades for a period varying from six months to two years. The report of the "Committee of Fifteen" of the National Education Association summarizes the tendency existing in 1895 when it states that, "with the right methods, and a wise use of time in preparing the arithmetic lesson in and out of school, five years are suffi-

cient for the study of mere arithmetic — the five years beginning with the second school year and ending with the close of the sixth year.

The Attempt to Eliminate Waste

The attempt to shorten the period of formal instruction in arithmetic has had its effects upon the methods of teaching as well as upon the arrangement of the course of study. The presence of a large number of children who leave school by the seventh year, the example of a varied European practice, the overcrowded curriculum, — all these have combined to suggest a shortened treatment of arithmetic. Hence economy, through the elimination of obsolete and unimportant topics in the course of study and through better methods of instruction, has become a pressing matter. Its influence on method is obvious.

It has focused attention upon "teaching method" and given it an increasing importance in the eyes of mathematics teachers. Specifically, it has tended to reduce the amount of objective work, to eliminate the explanation or rationaliza-

tion of processes which in life are done automatically; it has made teachers satisfied with teaching one manner of solution where, before, two or three were given; it has laid the emphasis upon utilizing old knowledge in new places, rather than on acquiring new means.

Delay in Beginning Formal Arithmetic Teaching

The tendency toward delay in beginning formal arithmetic instruction is to be explained in terms of several causes. Under a regimen where complicated and obsolete problems, difficult of comprehension, were common in elementary school tests, it was natural that teachers should believe that arithmetic is too difficult a subject for young children and that better results could be obtained if the subject were not commenced till the children were more mature.

This belief persists even after the curriculum is purged of all obsolete and over-complex materials, and has become a modern course of study with materials well within the comprehension and interest of primary children.

The Incidental Method of Teaching

The by-product of this belief is that any arithmetic taught during these first few years should be taught "incidentally," as a chance accompaniment of other studies. Only after one or two years of incidental work should the formal arithmetic instruction be given. This "incidental" method of teaching beginners is difficult to estimate. It has been so variously treated that a comparative measure of its worth is difficult to obtain. The contention that children who are taught incidentally for two years and systematically for two years more have at the end of four years of school life as good a command of arithmetic as those who have had a systematic course through four school years, is difficult to substantiate or deny on scientific grounds. Sometimes "incidental" teaching required by the course of study becomes "systematic" in the hands of the teacher. Sometimes the two years of "systematic" teaching which follow the incidental teaching mean far more than two years, since the teachers, in order to catch up,

give more time and emphasis to the subject than the relative time-allotment of any general schedule would seem to warrant. Such have been the facts frequently revealed by a class-room inspection that penetrates beyond the course of study, the time schedule, and regulations of the school board.

Reactions against the Plan of Incidental Teaching

In the lack of specific comparative measures of the worth of such methods of instruction, there is a growing conviction (1) that beginning school children are mature enough for the systematic study of all the arithmetic that the modern course of study would assign to these grades; (2) that, considering the quantity and quality of their experiences, they can think or reason quite as well as memorize; and (3) that what the school requires of the child can be better done in a responsible, systematic manner than by any haphazard system of "incidental" instruction.

These reactionary attitudes by no means imply a return to "systematic" teaching of arithmetic in the first two school years, nor to such formal

methods as were previously employed. Other grounds forbid. The crude, uninteresting memoriter methods of the past have gone for good. Objective work, plays, games, manual activities make arithmetical study easier and more efficient. Indeed, these newer methods have been a large factor in convincing teachers that children have the ability to master the first steps in arithmetic during the first two years. Regardless of this change in professional belief, it is a fairly general opinion that arithmetic should not be thrown upon the school-beginner along with the other heavy burden of learning to read. Learning the mechanics of reading is quite the most important part of the first school year, and the addition of the difficulties of another language — for such number is — would overburden and distract the child. Hence a common-sense distribution of burdens and tasks, regardless of questions of child maturity, would delay the formal and systematic study of arithmetic a half or whole school year, little reliance being placed upon previous "incidental" acquisitions.

Logical and Psychological Types of Arrangement

There are other problems of method less concerned with the time for beginning the study, or with the span of school life to be given to it. These deal with the arrangement of sub-topics within the course of study, or with the manner of progression from one aspect of arithmetical experience to another. I refer now to the various methods of planning the work in arithmetic from grade to grade, of which the "Grube method" and the "spiral" methods are types.

The methods that have been employed in the United States for the arrangement or ordering of topics within the course of study have varied considerably from time to time, but all these variations may be grouped around two types: (1) The "logical" types of arrangement, and (2) the "psychological" types of arrangement. If the course of study proceeds primarily by units that are characteristic of the mathematics of a mature adult mind, the type may be said to be "logical." If the course of study proceeds primarily by units that are characteristic of the

manner in which an immature child's mind approaches the subject, then the type may be said to be "psychological." The dominant arrangements have been "logical" up to within the past two decades. The older text-books taught "notation and numeration" rather thoroughly, then proceeded to a fairly adequate mastery of "addition," then to "subtraction," and so on. Such an arrangement is distinctly "logical." So also was the later "Grube method," which progressed by numbers rather than by processes.

The courses of study which have been most familiar to us in the past decade have used the "concentric circle" or "spiral" methods of arranging the sub-topics of arithmetic. These arrangements are "psychological" in type. They are attempts to give a systematic order of mastery which shall approximate the child's order of need in knowing. Here the first mathematical facts and skills taught are those that the child first requires, regardless as to whether they employ integers or fractions, additions or divisions. A little later, he deals with the same subjects and the same numbers in more complicated

manipulation and in more extended application. The field is re-covered, as it were, by ever widening circles or by an enlarged swing of the "spiral" progression.

Estimates of Worth

The older "logical" plans are thorough and definite in their demands; the teacher always knows just what he is about. But such a system of procedure is unnatural and remote from the child; it lacks appeal and motive. The child pursues the subject as a task laid down for him, not as an answer to his own curiosities or necessities. The newer psychological plans meet the different levels of child-maturity effectively; they are nearer the natural order of acquiring knowledge. The difficulty with all psychological arrangements is that the teacher cannot readily remember what the child has and has not been. The supervisor, too, finds it hard to locate responsibility for the teaching of definite arithmetic sub-topics. As orders of teaching they are psychologically natural but administratively ineffective.

The Present Mixed Method of Procedure

The result is that, to-day, the two types of arrangement are modifying each other and giving a mixed method, partly "logical" and partly "psychological." That line of least resistance in which the children study arithmetical facts and processes with greatest success is modified by definite demands that topics — *e.g.*, addition — be mastered thoroughly "then and there." The method is partly "topical" and partly "spiral." The child in the second grade may have a little of all the fundamental processes, a few simple fractions, and United States money; but just there he will be held definitely responsible for a very considerable number of the addition combinations. The pupil may have had fractions in every grade, but the fifth grade will be responsible for a thorough and systematic mastery of the same. Such is the mixed method of arrangement which is to-day prevalent in American schools.

V

THE DISTRIBUTION OF OBJECTIVE WORK

Objective Teaching is Generally Current

THE use of objects in teaching arithmetic is current in the elementary school. Particularly is this true in the lowest grades of the school, in primary work. It may be said that there is a very large quantity of objective teaching in the first year of schooling and that it decreases more or less gradually as the higher grades are approached. By the time the highest grammar grades are reached, the use of objects has reached its minimum.

The teaching of arithmetic prior to the middle of the nineteenth century was little associated with object teaching. That is to say, the general practice of instruction was non-objective. The use of objects in giving a concrete basis for abstract arithmetical concepts and for memoriter manipulations, seems to have gained its initial

hold on the schools through the introduction of Pestalozzian methods of teaching. The later introduction of school subjects requiring objective treatment, such as elementary science, nature study and manual training, fortified the previous movement and gave it considerable enlargement. Together these two movements established the respectability of objective teaching in arithmetic. School-room experience quickly gave it an empirical sanction. It remained for the modern psychological movement in education to give it a scientific sanction, and to refine its uses.

Its Distribution is Crudely Gauged

It is quite fair to say that the use of objective work decreases more or less gradually from the first to the last year, the underlying assumption being that the use of objects has a teaching value that decreases as the maturity of the pupils increases. Current practice does not proceed far beyond the application of the simple and somewhat crude psychological statement that the youngest children must have much objective

teaching, the older less, the oldest least of all. The lack of a more refined analysis of the worth of object teaching necessarily leads to some neglect and waste.

If a new topic enters late into the course of study, as in the case of square root, the subject is not so well taught because of the current prejudice or tradition against the use of object teaching in the higher grades. On the other hand it is also probable that the teaching of addition is often accompanied by wasted time and energy simply because lingering over objects in the lower classes is the current fashion.

Tendency toward a More Refined Correlation of Object Teaching with Particular Immaturity

Reform in the direction of a more refined and exact use of object teaching has already appeared in the treatment of fractions and mensuration, where, regardless of the increased maturity of the children studying these topics, a large amount of objective method is utilized. This is a considerable departure from the slight objective treatment of other arithmetic topics

taught in the same grades. Such exceptional practices suggest that the novelty of an arithmetic topic is the condition calling for objective work in instruction. It is immaturity in a special subject or situation which determines the amount of basal objective work. The correlation is not with the age of the pupil, but with his experience with the social problem or subject in hand. It is of course true that the younger the student is, the greater the likelihood that any subject presented will be novel and strange. Only in this indirect manner does the novelty of subject matter coincide with mere youth as an essential principle in determining the need of objective presentation. The naïve assumption of the older enthusiastic reformers that objective work is a good thing psychologically, one of which the pupil cannot have too much, is by no means the accepted view of the new reformer. With the latter, objective presentation is an excellent method at a given stage of immaturity in the special topic involved; but it may be uneconomical, even an obstacle to efficiency, if pushed beyond.

The Movement Supported by both Scientific and Common-Sense Criticism

There is, then, a certain coincidence of the scientific criticism of the psychologist and of the common-sense criticism of the conservative teacher, who look suspiciously upon a highly extended object teaching. The teacher, on grounds of experience, says that too much objective teaching is confusing and delays teaching. The psychological critic says it is unnecessary and wasteful. The result is that, in these later days, the distribution of objective work has changed somewhat. More subjects are developed in the higher grades through an objective instruction than before. Perhaps no fewer subjects in the lower grades are presented objectively, but the extent of objective treatment of each of these has undergone considerable curtailment.

VI

THE MATERIALS OF OBJECTIVE TEACHING

The Indiscriminate Use of Objects

THE existing defects in objective teaching are not restricted to a false placing or distribution. The quality of the teaching use of objects is likewise open to serious criticism. Object teaching is a device, so successful, as against prior non-objective teaching, that it has come to be a standard of instruction as well as a means. As long as objects — any convenient objects — are used, the teaching is regarded as good. Given such a sanction, the inevitable result is an undiscriminating use of objects. The process of objectifying tends not to be regulated by the needs of the child's thinking life; it is determined by the enthusiasm of the teacher and the materials convenient for school use.

The Artificiality of Materials Utilized

The first fact which is noted in observing objective teaching is the artificiality of the materials

employed. Primary children count, add, etc., with things they will never be concerned with in life. Lentils, sticks, tablets, and the like are the stock objective stuff of the schools, and to a considerable degree this will always be the case. Cheap and convenient material suitable for individual manipulation on the top of a school desk is not plentiful. But instances where better and more normal material has been used are frequent enough in the best schools to warrant the belief that more could be done in this direction in the average classroom. The "playing at store," the use of actual applications of the tables of weights and measures are cases that might be cited.

Narrowness in the Range of Materials

The materials used are not only more artificial than they need be, but too restricted in range. As has already been said, the types of material capable of convenient and efficient use in a schoolroom are not numerous. But the series can and ought to be extended. More forms of even the artificial material should be used, thus minimizing the danger of monotony. The blame

for the narrow range of materials used falls partly on school boards who do not vote a sufficient allowance for teaching materials to primary teachers; partly on teachers who do not exercise sufficient ingenuity in devising new forms of objects, or who do not show the vigor requisite to a shift from one material to another; and partly on the supervisory staff which has neither been insistent upon, nor sensitive to, the need of a more interesting range of objective stuffs.

Inadequate Variation of Traditional Materials

Even the narrow range of materials in general use might be better employed than it is. There is, of course, a distinct tendency to vary the objects, merely because a child gets tired of one kind as a material. But a different quality of variation is required when the pupil is to derive abstract notions from concrete materials. It is too frequently the case that the teacher will treat the fundamental addition combinations with one set of objects, *e.g.*, lentils. In all the child's objective experience within that field there are two persistent associations — "lentils" and "the

relation of addition." The accidental element is thus emphasized as frequently as the essential one and, being concrete, has even a better chance to impress itself. A wide variation in the objective material used would make teaching more effective, particularly with young children.

The Restricted Use of Diagrams and Pictures

The nature of the materials proper to objective teaching has likewise been too narrowly interpreted. Objective teaching has meant, almost exclusively, instructing or developing through three-dimensional presentations. There is a wide range of two-dimensional representations which have been neglected, but which for all the psychological purposes of education have as much worth as so-called objects. I refer here to the use of such material as pictures. Such quasi-objective material has been little used by teachers save as it appears in textbooks. Even the textbook writers have not used pictures with a deep sense of their intrinsic worth. They are printed as a mere substitute for objects in a period when objects are popular pedagogical materials. The

geometric figure or diagram has had a slight use with both the teacher and the textbook writer. Its most frequent use has been in treatments of mensuration. There are, of course, obvious disadvantages to pictures and diagrams. The things represented in them are not capable of personal manipulation by the child in the ordinary sense. But they have a superiority all their own. They offer a wider, more natural, and more interesting range of concrete experiences.

Plays and Games in Object Teaching

There are other curious phases of narrowness in the current pedagogical interpretation of what constitutes a concrete or objective experience. It will be noted that visual objects are the ones generally employed and that they are generally inanimate objects. Of late there has been some tendency to use hearing and touch in giving a concrete basis to teaching. Advantage is taken of the social plays of children, and their games with things. Here the children themselves, and their relations and acts are the experiences from which the numerical units are obtained. With

some of the best teachers in the lowest grades it is no longer unusual to see children moving about in all sorts of play designed to add reality to, and increase interest in, number facts.

The Lack of Unity in the Use of Objects

The conservative teacher's use of objects is hopelessly artificial and lacking in unity. If he brings a series of objects into the development of a single topic, they have little relation to each other, and they represent no actual grouping. Their sole connection with one another is that they exemplify the same abstract arithmetical truth. Beans, cardboard squares, and shoe-pegs may all be employed in the same lesson. The progressive teachers offer more logical unity in their materials. To "play at store," to utilize games, to deal with things within a picture, is to bring the concrete materials into the classroom with a more nearly normal setting. It is in no small measure due to this better use of material that the progressive teacher is gaining power throughout the elementary grades.

VII

SOME RECENT INFLUENCES ON OBJECTIVE TEACHING

THE wasteful use of objective teaching in the lowest grades has undergone some correction. The sheer enthusiasm of the modern reformer is partly responsible for this modification of conservative practice. When did single-minded men ever keep within bounds? In our social economy the defense of the radical is found in the fact that other single-minded men are conservatives. Men at one extreme need to be overcome by like men at the other. But the check of one enthusiast on another is not always perfect. Other contributory causes make it easy to go to unfortunate extremes not easily corrected.

The Influence of Inductive Teaching

Inductive teaching has been one of several movements affecting objective teaching. The effort of teachers to escape the slavishness of

mere memoriter methods and to approximate real thinking led to the introduction of inductive teaching. Necessarily objective teaching became more or less identified with the new movement and was influenced by it. So, it has been said of objective work in arithmetic as it has been said of laboratory work in the sciences, that such instruction is a method of "discovery" or "rediscovery." Such an alliance has had its beneficial effects upon objective teaching; it has redeemed it from the aimless "observational work" of an earlier "objective study." But in the teaching of arithmetic, at any rate, it has confused an objective mode of presentation with a scientific method of learning truth, two activities having a common logical basis, but not at all the same. Under the assumption that the "development" method is one of "rediscovery," the tendency is to give the child as complete a range of concrete evidences as would be necessary on the part of the scientist in substantiating a new fact. The result is, that long after the child is convinced of the truth, say that 4 and 2 are 6, the teacher persists in giving further objective illus-

trations of the fact. The child loses interest in the somewhat monotonous continuance of objective manipulations, and the teacher has naturally wasted time and energy. If the fact or the process that the teacher wishes to convey can be transmitted with fewer objective treatments (the authoritative treatment of the teacher counting for something in school, as authority counts everywhere), then it is unnecessary to exhaust the objective treatments of a numerical fact. Inductive teaching and learning are not equivalent to inductive discovery; and to hold them identical is necessarily to overdo the use of objects in teaching.

The Movement for Active Modes of Learning

Another modern movement in teaching method which has had a conspicuous effect on objective teaching is the Froebelian demand for "self-activity" on the part of the child. The recent favor enjoyed by manual training, nature study, self-government, and other active phases of school life is indicative of the sway of this doctrine. Its influence has forced the introduction of new sub-

jects and changed the manner of presenting the older subjects of the elementary curriculum. Arithmetic has responded along with the other studies, and an active use of objects by the children themselves is found in increased degree.

There was a time when objective work in the schools was a passive matter so far as the child was concerned. Any active manipulation of the objects that might be required was cared for by the teacher, the child being merely a passive observer. This is much less the case than formerly, the influence of "self-activity" having entered with contemporaneous pedagogy. The present situation is one where the child sometimes merely observes objects and sometimes actually handles them.

At present, then, we have about the same range of methods employed in teaching arithmetic as in teaching science. At one extreme the teacher himself demonstrates by the help of objects in the presence of the class, and records the relations in appropriate arithmetical symbols, the class being in the position of interested spectators of a process. At the other extreme the

teacher puts the material on the desks of the children and, with a minimum of instruction in advance, directs them toward the desired experiences and conclusions.

The Abbreviated Use of Objects

As might be expected, there has been some reaction against the influences emerging from inductive or developmental teaching and active modes of learning. To a considerable degree the reactionary influence expresses itself in the abbreviated use of objects in presenting a mathematical relation, process, or manipulation. One mode of abbreviation will suffice as an example.

There are two methods of relating the symbols and processes of arithmetic to the actual relations among objects. For convenience these may be called the methods of "parallel correspondence" and of "final correspondence."

The Method of Parallel Correspondence

The method of "parallel correspondence" is generally used in the development of all the simpler combinations or processes of arithmetic.

In learning to count, the child sees the first object and says the symbolic "one," sees the second object, and says the symbolic "two." Again in addition, he sees "ten," and writes the symbol 10; sees "six," and writes 6; sees the whole as sixteen, and writes 16; then summarizes the work in the form $10+6=16$. Each stage in the symbolic process is noted in connection with objects. This, the method of "parallel correspondence," is the more current method of using objects.

The Method of Final Correspondence

The method of showing a "final correspondence" of result between objective manipulation and symbolic manipulation is much less frequently used. It is used with more complex processes than those mentioned above, in connection with column addition or "borrowing" in subtraction. It is a mode of object teaching used in place of the usual "explanation" or "rationalization" which attempts to explain what is simply a correspondence between the manipulation of a series of facts and the manipulation of a se-

ries of symbols. Under this method the teacher usually tells the child directly how to perform the process in the conventional manner, no special explanation being given. Then a case involving the actual use of objects is considered, and this result is compared with the result obtained by the symbolic manipulation. One or two such cases suffice to convince the pupil that the authoritative mode is true to nature. This method of "final correspondence" in the use of objects represents a new and restricted, but increasing, tendency.

VIII

THE USE OF METHODS OF RATIONALIZATION

The Tendency toward Rational Methods

Some of the marked changes which have occurred in the methods employed for the presentation of number to children have already been mentioned in connection with the objective teaching of arithmetic. The main tendency to be noted is that objective instruction, which has been used as a mere device of illustration, becomes the first step in inductive or developmental teaching. It is subsumed under a more inclusive method. The change is significant, for it is a symptom indicating that mathematical teaching is becoming less dominantly memoriter and more rational.

The Era of Direct Instruction and Drill

Several decades ago it was not at all unusual for the bare facts of arithmetic to be given to the child by the teacher without much attempt at

providing a basis in the pupil's own experience. The teaching was "direct," the teacher's attention being focused on getting the fact from his own mind into the child's mind, the whole emphasis being placed upon the problem of transmission and the subsequent difficulty of retention. In so far as objects and illustrations were used, they were merely incidental and reinforcive. They did not constitute any basic body of personal experience by which the child was to seize a concept, relation, or process to be handled thereafter through symbols and conventional forms. Under such a system of instruction, still too widely prevalent, the child had to memorize outright, without any concrete basis for his belief, tables of addition, multiplication, etc., and the rules for solving various types of problems. It was outright memorization for which little vital motivation was provided. In insuring retention the teacher therefore relied, not upon interested and varied impressions, but upon the number of verbal repetitions. "Drill" was characteristic of this era in teaching.

Indirect Teaching as a Rational Method

Under the influence of the inductive or developmental method of teaching, the emphasis on the repetitional memorization of number facts and processes is reduced. Teaching now becomes "indirect" rather than "direct"; the child learns through his own experience rather than through the statement of book or teacher. Here the child's own thought and activity, not the teacher's, are conspicuously central in the teaching situation. The teacher stimulates the child into action; he suggests, guides, corrects, does everything in fact save obtrude his authority and opinion into the child's interpretation. The child's activity gives him many vivid and varied impressions of the subtraction combination or other relation. When he has *found* the fact, he has already learned it! Further drill or review is not primary, but simply supplementary — a further guarantee of the persistence of the impression. Even the spirit of such supplementary drill and review is, in these days, something different from a monotonous repetition of the same words; it is a reimpression

or review of the essential fact in many varied and interesting forms.

Interest as a Factor in Methods of Rationalization

It is perfectly natural that, in shifting the teacher's attention from his own activities to those of the children, the interest of the child should be considered in increasing degree. If the child is to learn directly, with a maximum use of his initiative, it is absolutely essential that the teacher should provide some motive. This implies that the child is to be interested in some fundamental way in the activities in which he is to engage. Instead of thumbing the fundamental facts with his memory, in an artificial and effortful manner, "sing-songing" the tables rhythmically, so as to make dull business less dull, the child studies the arithmetic involved in his own life, for the modern teacher brings the two together. The number story, the arithmetical game, playing at adult activities, constructive work, measuring, and other vital interests of the child and community life become increasingly the basis

of instruction in number. Such is the pronounced tendency wherever the movement is away from the traditional rote-learning or drill.

Of course there is a slight tendency in American elementary schools, where a soft and false interpretation of the "doctrine of interest" is gospel, to teach only those things that can be taught in an interesting fashion. But this tendency is less operative in arithmetic than in other subjects. Here the logical interdependence of one arithmetical skill on another has quickly pointed the failure of such a haphazard mode of instruction.

The Reaction against Rationalization

There is, however, in "advanced" as well as in reactionary quarters, a revolt against the tendency to objectify, explain, or rationalize everything taught in arithmetic. On the whole it is a discriminating movement, for this opposition to "rationalization" in arithmetical teaching, and in favor of "memorization" or "habituation," bases its plea on rational grounds, mainly derived from the facts of modern psychology.

It is specifically opposed to explaining why "carrying" in addition, and "borrowing" in subtraction are right modes of procedure. These acts are to be taught as memory or habit, inasmuch as they are to be performed by that method forever after. To develop such processes rationally or to demand a reason for the procedure once it is acquired, is merely to stir up unnecessary trouble, trouble unprompted by any demands of actual efficiency.

Four Principles for the Use of Rationalization

A study of the actual arithmetical facts upon which this opposition expresses itself suggests the four following general principles as to the use of "rationalization" and "habituation," as methods of mastery: (1) Any fact or process which always recurs in an identical manner, and occurs with sufficient frequency to be remembered, ought not to be "rationalized" for the pupil, but "habituated." The correct placing of partial products in the multiplication of two numbers of two or more figures is a specific case. (2) If a process does recur in the same manner,

but is so little used in after life that any formal method of solution would be forgotten, then the teacher should "rationalize" it. The process of finding the square root of a number illustrates this series of facts. (3) If the process always does occur in the same manner, but with the frequency of its recurrence in doubt, the teacher should both "habituate" and "rationalize." The division of a fraction by a fraction is frequently taught both "mechanically" and "by thinking it out." (4) When a process or relation is likely to be expressed in a variable form, then the child must be taught to think through the relations involved, and should not be permitted to treat it mechanically, through a mere act of habit or memory. All applied examples are to be dealt with in this manner, for such problems are of many types, and no two problems of the same type are ever quite alike. These laws will, of course, not be interpreted to mean that no reason is to be given a child in a process like "carrying" in addition. The reason is not essential to efficient mastery, but it may be given to add interest or to satisfy the specially curious.

The Substantiating Psychology

The theoretic basis which seems to underlie such a statement of general principles is derived from functional psychology. Memory and reasoning are not separate functions; they are interdependent; but we mean differently by these terms because they have distinguishable emphases. It may be said that memory as a function is efficient only in the face of familiar situations where, if the association is present at all, the response is quick and precise. In the face of new situations it is incapable of accurate response. Reason is slow and uneconomical in action, but it is the only efficient method of arriving at the essential nature of a problem largely unfamiliar. It would be wasteful to meet many of the necessary events of life with a purely reasoned reaction. It would be too devious and deliberate in reaching conclusions.

Rationalization as a Substitute for Object Teaching

There is a sense in which all proof through objects is a type of rationalization, but we do

not ordinarily so consider it. Such a mere "correspondence with the objective facts" is sufficiently different from the method of "explaining a new fact in terms of previously acquired facts" to warrant a separate classification. Were it not that the methods are sometimes interchangeable in developing arithmetical truths, they would not need to be mentioned here. A citation will make the point clear. In teaching the multiplication tables, the combination "six twos are twelve" may be taught as a direct objective fact, as when six pupils with two hands each are shown. On the other hand the same fact of multiplication may be taught as a "derived fact," as when six twos are added in a column and make twelve. They are both rational methods of proving that six twos are twelve. One method shows it objectively, the other through the use of well established addition combinations viewed as multiplication. Such a rational mode of "deriving" multiplication is used more frequently perhaps than the objective method.

IX

SPECIAL METHODS FOR OBTAINING ACCURACY, INDEPENDENCE, AND SPEED

Supervision of Learning after First Development

It is not alone the first stages in the acquisition of an arithmetical process which have received attention in the re-organization of teaching methods, though, to be sure, the problem of first presentations has in recent decades been given the most attention. More and more, the American tendency is to watch every step in the learning process, to provide for all necessary transitions, and to safeguard against avoidable confusions. It might be suggested that the intermediation of the teacher at every step in the child's work might destroy the pupil's initiative and independence. Apparently, however, those who are deeply interested that the child should not be permitted to fall into the errors which unsupervised drill would convert into habits, are fully as cautious to provide steps for forcing the

child to assume an increasing responsibility for his own work. The distinction made is that an over-early independence is as fatal to accurate and rapid mathematical work as an over-delayed dependence.

The Use of Steps, or Stages, in Teaching

Some of these serial treatments or related stages of method to which reference has been made may be cited. Necessarily only the more important are mentioned. In indicating certain clean-cut steps or processes, there is no intent to convey the idea that these stages are fixed or conscious matters. The statement is merely indicative of the habitual tendency of the average practitioner with an implied theory. As will be readily evident, there is no assumption that such a formal, classified, theoretic statement of stages is a conscious possession of the teaching staff in general. Again, in actual school work there are many types of variation from the characteristic modes here suggested. Always the steps overlap; frequently they are extended, abbreviated, or omitted. But the statement represents, in a very

real way, the trend of underlying theory, whether conscious or merely implied.

Stages in the Presentation of Problems

One of the specific controversies much argued in the primary school concerns the medium through which arithmetical examples and problems shall be transmitted to young children. There are three typical ways in which a situation demanding arithmetical solution may be brought to the child's mind: (1) The situation when visible may be presented through itself, that is, objectively. (2) The situation may be described through the medium of spoken language, the teacher usually giving the dictation. (3) The situation may be conveyed through written language, as when the child reads from blackboard or text. Inasmuch as objects are a universal language, no difficulty arises through this basic method of presentation. It is when a language description of a situation is substituted for the situation itself that difficulty occurs. The child might be able to solve the problem if he really understood the situation the language was

meant to convey. Owing to the difficulty that primary children have in getting the thought out of language, it has been urged that problems in any unfamiliar field should be presented in the following order: (1) Objectively or graphically; (2) when the fundamental idea is grasped, through spoken language; and (3), after the type of situation is fairly familiar, through written or printed language. It is seriously urged by some teachers that no written presentation should be used in the first four grades. Such an extreme tendency would practically abolish the use of primary text-books. There are many exceptional teachers who do not put a primary text in the hands of children at all. Such a tendency is increasing. Particularly is this true among primary teachers in the schools of the foreign quarters of large cities. Accurate communication through the English language is always more difficult here. Hence, the period of objective teaching is necessarily prolonged, dependence on the "number stories" told by the teacher extended, and the solution of written problems much longer delayed than elsewhere.

An Opposite Method in Presenting Examples

The situation is somewhat different, almost the opposite in fact, when "examples" rather than "problems" are presented, meaning by "example" a "problem" expressed through the use of mathematical signs. It is easier to present "examples" in written form on blackboard or in text than it is to dictate them orally. This obviates the necessity of holding the examples in mind during solution. The permanence of the visual presentation saves the re-statement frequently necessary in oral presentation. Hence it is a common practice to supply the youngest children with mimeographed or written sheets of "examples." It is with older children, or with younger children at a later stage in the mastery of a typical difficulty, that oral presentation of examples is stressed. Then we have that type of work which is called "mental" or "silent" arithmetic.

Better Transitions from Concrete to Abstract Work

There is some tendency toward the provision of better transitions from the objective presenta-

tion of applied problems to the symbolic presentation of abstract examples. Behind all uses of objective work is the belief that it is a mere foundation for more rapid and efficient abstract work. Objective teaching is fundamental, but purely preparatory. The child ought to pass from objects and sense-impressions, through images of various degrees of abbreviation, to symbols and the abstract concepts for which they stand. But in American practice a sharp jump is usually made from concrete objects to abstract symbols. The transition through adequate transitional imagery is not made. Wherever psychological influences are directly at work in the schools, there is a minority movement favoring a better transition. The nature of such transition is scarcely reasoned out as so much psychological science, but is the accompaniment of a widening professional movement for the enlarged use of pictures, diagrams, number stories, and the like. A critical examination of the various means of presenting arithmetical situations would order them as follows in making the transitions from objective concreteness to

symbolic abstraction: (1) Objects, (2) pictures, (3) graphs, (4) the concrete imagery of words, (5) more abstract verbal presentations, (6) presentations through mathematical symbols. No such minuteness of adjustment is apparent in existing methods, though it might seem desirable in teaching young children. At any rate, it would be more effective than an unreasoned traditional procedure full of over-emphasis and omission.

The Child's Four Modes of Work

We have thus far discussed merely the teacher's activity in instruction. We have to note the graded requirement made in the child's own activity. What is the existing custom with reference to the manner in which children are required to solve the problems or examples presented to them? There are four typical ways in which the child does his work, the names of which are derived from the differentiating element: (1) The "silent" method, otherwise spoken of as "mental arithmetic," "arithmetic without a pencil," etc. (2) The "oral" method where the child works

aloud, that is, expresses his procedure step by step in speech. (3) The "written" method where the child writes out in full his analysis and calculation. (4) The "mixed" method where the child uses all three of the previously mentioned methods, in alternation, as ease and efficiency may require.

The Worth of these Modes

The worth of these four methods of work is necessarily variable. The rapidity of the "silent method" with simple figures is obvious. The "silent method" and the "mixed method" (which is more slow but more manageable with complex processes and calculations) are the two methods normally employed in social and business life. The purely "oral" and "written" methods, with their tendency toward analysis and calculation fully expressed in oral or written language, are highly artificial. They are valuable as school devices for revealing the action of the child's mind to the teacher so that it may be corrected, guided, and generally controlled. The present tendency is toward an over-use of these methods

and toward an under-use of the other two, more particularly the "mixed" method. It would seem that there is little conscious attempt to make certain that the child moves from full oral or written statement to the judicious application of the more natural "silent" and "mixed" methods. It may be that full oral and written statements of work have seriously hampered the right use of the more natural methods of statement.

The Traditional Quarrel between "Mental" and "Written" Arithmetic

One conspicuous traditional quarrel in the schools is between the "silent" and the "written" methods. Up to within a decade or so ago, "silent" or "mental" arithmetic was much over-emphasized, being carried to absurd extremes. The reaction that followed was equally extreme in its emphasis on the "written" method. There are signs now of a more rational use of the two as supplements of each other.

The order in which different statements of arithmetical work should come has also been a subject for pedagogical argument. The usual

order, due to the fact that first treatments of a topic are simple both in the steps of reasoning and the calculations involved, has been "silent" arithmetic followed by "written" arithmetic. A more recent order has been: (1) "silent" (2) "written" (3) "silent" — a much superior serial order, though by no means an accurate statement of a perfect procedure. The fixed treatments necessitated by text-books have made teaching method arbitrary in its steps, here as elsewhere. It is altogether probable that many simple calculations or analyses can be done "silently" ("mentally") from the beginning; that others require visual demonstration, but once mastered can thereafter be done without visual aids; that still others will always have to be performed, partially at least, with some written work. It is purely a matter for concrete judgment in each special case, but the existing practice scarcely recognizes this truth. The result is that many problems are arbitrarily done in one way, and it is too frequently the uneconomical and inefficient way that is used.

The Transition from Development by Teacher to Independent Work by Pupil

It is well to recall that in all these efforts to control the child's activity, there is a tendency to leave the child over-dependent upon the teacher. It is vitally important that a child should be kept free of any error which unsupervised drill would fix into a stubborn habit; but it is likewise important that the child should acquire some self-reliance. While not always clearly defined, there is a distinct tendency in the direction of releasing the teacher's control of the child. A characteristic practice would be one in which the teacher's work with the child would pass through various stages, each one of which would mark a decrease in the control of the process by the teacher and an increase in the freedom of the child to do his example, or problem, by himself.

Four Characteristic Stages of the Transition

One characteristic series of stages quite frequently used in the presentation of a single topic

in arithmetic, say "carrying" in addition, is the following: (1) The teacher performs the process on the blackboard in the presence of the class, the children not being allowed to attempt the process by themselves until after the process is clearly understood from the teacher's development. (2) The children are then allowed to perform the process upon the blackboard, where it is exceedingly easy for the teacher to keep the work of every child under his eye. An error is caught by a quick glance at the board and immediately corrected before the child can reiterate a false impression. (3) Other cases of the same type of example are assigned to the children at their seats where they work upon paper, still under the supervision of the teacher — a supervision which is less adequate, however. (4) The same difficulty, after the careful safeguarding of the previous stages, is then assigned for "home work," where the child relies almost completely upon himself. Once more it is necessary to suggest that these stages are merely roughly implied in the variations of existing practice.

Special Methods of Attaining Speed

Most of the methods discussed in this chapter have had as their sanction the attainment of accuracy in thinking and calculating. Some efforts to insure independent power on the part of the child have already been noted. But nothing has been said of the effort to add speed to accuracy in getting efficient results. Such special efforts have been made. These efforts may be classified into two groups: (1) Those aiming to quicken the rate of mental response. (2) Those aiming at short-cut processes of calculation.

Typical of the first are (*a*) the use of an established rhythm as the child attacks a column of additions; (*b*) the device of having children race for quick answers, letting them raise their hands or stand when they have gotten the answer; (*c*) the assignment of a series of problems for written work under the pressure of a restricted time allotment for the performance of each. These and similar devices are much used in the schools. They are open to the objection that they quicken the rate of the better students, but foster

confusion, error, and discouragement among the less able children, thus actually hindering speed.

The various shorter methods which represent an effort to reduce the number of mental processes required are usually not of general applicability, and consequently have not attained general currency in the elementary schools which aim to teach merely one generally available and effective method even though it requires more time, special expertness being left to later development in the special school or business which requires it.

The Relation of Accuracy to Speed

It has come to be quite a common opinion among teachers that the fundamental element in rapid arithmetical work is certain and accurate calculation. If pupils know their tables of combinations and are sure of each detail of calculation, there is no confusion or hesitancy; speed then follows as a matter of course. This belief, as much as anything else, explains why the lower schools have developed few special means for attaining speed other than those mentioned.

X

THE USE OF SPECIAL ALGORISMS, ORAL FORMS, AND WRITTEN ARRANGEMENTS

THE methods of teaching arithmetic are influenced not only by the aims of such instruction, but by the peculiar nature of the matter taught. The use of special algorisms, temporary algoristic aids or teaching "crutches," oral and written forms of analysis, are of considerable moment in determining the difficulties and therefore the methods of teaching. Their condition and influence will need to be given some slight notice.

The Traditional Nature of Algorisms and Forms

The algorisms and forms used in the American schools are those that have been determined by social and educational traditions. It is probable that wide social practice has largely determined the traditions, though it must be admitted that the traditions of text-book makers have also given it form. In consequence the ruling school

tradition in the matter of algorisms does not always coincide with current community practice. It is probable that the various modes of computing interest, given by the average arithmetic of ten years ago, were once current in the business world. These methods have changed somewhat, and the school form of computation has not always been changed to accord. Such misadjustments between the forms used in school and those used in daily life are not numerous, but they are more frequent than they ought to be.

The Number of Algorisms Used

The use of various algorisms for a single process is not very frequent. There is a fairly general prevalence of a single algorism for a single process among American teachers. Certain striking exceptions are to be noted in connection with subtraction and division, where the so-called "Austrian" methods are being brought into competition with the traditional modes of the American school. It may be said, however, that even when two distinct algorisms are in contemporaneous use in a school system, the teachers

are usually careful to employ only one algorism with a given child. Even when a child moves from a school using one kind of algorism to a school using another, the tendency is to allow him to follow his own method.

Reform in the Use of Algorisms

The tendency toward the substitution, duplication, and modification of existing algorisms is inconsiderable, and very recent. The introduction of the "Austrian" algorisms, already mentioned, is perhaps the most conspicuous movement, having a very wide group of adherents. There are, however, a group of teachers and educational psychologists who are attempting to refine teaching methods so as to attain a greater economy and efficiency in the learning process. These are responsible for a movement toward the reform of existing algorisms. The movement expresses itself in a number of ways, — it offers new forms, modifies others, and aims to bring a larger similarity and consistency into algorisms employed in the various stages of the same process. Its function is always to simplify for the child and

thus to increase the practical efficiency and the mental economy of his methods.

The Standard of Social Usage

One of the standards set, is, that as far as is consistent with economy, the algorism employed with greatest frequency in social life is to be preferred. If people add, subtract, and multiply with their figures arranged above and below each other in vertical form $\begin{array}{r}6\\\underline{3}\\9\end{array}$, then the vertical form is to be preferred to the horizontal method $6+3=9$ so largely used and imposed by textbook writers.

The Extended Use of Acquired Forms

To learn two forms for one thing, particularly when one has no sanction either in current use or on general grounds of psychological efficiency, is a waste. Hence there is an increasing disposition both in general practice and among the more critical to utilize a single form in as many places as possible. "Subtracting by adding" is merely using the same association and word form for both addition and subtraction. Hence only

one set of tables, instead of two, has to be learned. The meaning, the applicability, and the visual form of addition and subtraction are still different. Only the process of remembering and using the fundamental combinations is the same. But this is a large saving: "Dividing by multiplying" is an analogous situation, though not so much employed in American schools as "subtracting by adding." The most radical suggestion for utilizing a simplified common form is one in which these forms of division, as used in the tables $18 \div 6 = 3$, in short division $6\underline{)1832}$, and in long division $62\overline{)18325(}$, are reduced to one consistent form in all three cases, as $6\overline{|18}$, $6\overline{|1832}$, $62\overline{|18325}$. Such a simplification is urged in other situations. The movement has not passed far beyond theoretic acceptance, though several city school systems are trying the experiment, San Francisco being a notable instance.

The Use of "Crutches" or Temporary Algorisms

The use of special and temporary algoristic aids or learning "crutches" in mathematical calculation is one of the problems of method under

constant controversy. Teachers seem fairly evenly divided upon the question. Typical situations in which such "crutches" are used may be noted as follows: Changing the figures of the upper number in "borrowing" in subtraction; rewriting figures in adding and subtracting fractions. In the broad sense any algorism which is used during the teaching or learning process temporarily, to be abandoned completely later, is an "accessory algorism" or "crutch." The objections to their use lie in the facts: (1) that skill in manipulation is learned in connection with stages and forms not characteristic of final practical use; (2) that this implies, psychologically at any rate, the waste of learning two forms or usages instead of one; and, (3) that it decreases the speed with which mathematical calculation is done. If there is a drift in any direction, it is probably toward the abandonment of "crutches."

Full and Short Forms of Calculation

The division of opinion, which exists in connection with the temporary use of special algorisms or "crutches," likewise exists with reference

to the use of "full forms" and "short forms" of manipulation and statement. The temporary use of a "full form," in a case where a "short form" will finally be used, is similar to the employment of a "crutch." There is one important difference, however, which explains the relatively larger presence of temporary "full forms" than of "crutches." The "full form" is an accurate form which is used somewhere, — in a more complex stage of the same process or in some other process; the "crutch" is not. Thus: a "full form" in column addition with partial totals and a final total of partial totals, will be utilized in column multiplication, the "long division form" of doing "short division" (which is the fully expressed form of dividing by a number of one figure) will be utilized in division by numbers of more than one figure.

Forms of Analysis or Reasoning

The problem of form applies not alone to the algorism or special method of computation, but likewise to the special methods of reasoning used in determining the specific series of steps to be

taken in achieving the answer. In every problem the child solves, he must not only *decide what is to be done* (reason), but he must *do it* (calculate). There are forms of reasoning as there are forms of calculation. As any calculation may have several algorisms, the solution of a problem may be expressed in several forms. It is the latter difficulty which appears in the teacher's demands for "formal analysis" of problems. The analysis is usually required in full statement.

The Traditional Requirement of Full Formal Analysis

It has been a very general requirement in American schools that the child give a full oral or written statement of his analysis and computation. Formal statements have been demanded of the child as much on the side of reasoning as on that of calculation. One of the causes of this demand is found in the tendency of the teacher to encourage full statement by the child, merely as a revelation of his inner processes so that the source of error in results might be detected and the error eliminated. We have already noted this

predisposition of the teacher to call for full oral and written statements for purposes of control in the various methods designed to achieve and safeguard accuracy.

The Limitations of Full Formal Analysis

There is, however, a marked tendency away from formal analysis of arithmetic problems in the elementary school, just as there is a movement away from a formal deductive logic in the higher schools. Natural, genetic modes of thought are supplanting the unnatural, formal statement of steps. It is felt that while such full formal statements of reasoning and calculation may assist in the teacher's control, they may actually interfere with accuracy and rapidity on the child's part.

To write out each step in full often means giving an enlarged attention to factors that are merely touched and assumed in actual thinking. To delay the thought process, with attention held on a fully developed linguistic statement, whether oral or written, may be to distract from the chain of essential ideas or meanings that really solve

the problem. Frequently children lose the thread of thought midway of the process because of the necessity of dealing with the form side, and have to begin anew.

"Labeling" the Steps of Calculation

A conservative protest against the old formal expression of reasoned steps is found in omitting for the most part the linguistic statements dealing with the logic of the problem and merely "labeling" the numbers that occur in the calculation. This is a more restricted form of statement, much more used at the present time than hitherto. But it is still open to psychological objections that make the more scientific critics protest. There are many stages in a calculation where there is no association whatever with the concrete problem in hand. The concrete problem is studied, the decision is made that all the factors named are to be added. They are added, purely abstractly, and a number is given as the total. The result is then thought of in terms of the concrete problem in hand. A disposition to label each item in the addition may be ne-

cessary in the rendering of a bill, but it is a false and obstructing activity in the actual solving of the problem. The same situation exists where there are two or three processes to be utilized in series. Once the child has grasped his concrete situation and reasoned what to do — he may proceed to mechanical manipulation, never thinking of the concrete applications till he has finished.

So-called Accuracies of Statement

The protest mentioned above even goes so far as to attack the teacher's insistence upon certain so-called accuracies of statement. The case of 3 pencils at 5 cents would be expressed 3 times 5 cents = 15 cents. $3 \times 5 = 15$ would be demanded, and 5×3 not allowed at all. The protest is not against insistence on a proper order where "labeled" statements are used. The objection is made against the demand for the so-called proper order when abstract figures are related merely by signs. Where the child calculates symbolically, he sees the situation as one to be worked out by a purely conventional relation between

two numbers. For all practical purposes 5×3 will solve the situation quite as accurately as 3×5. The insistence on one, as opposed to the other, is a useless effort that cannot affect the result.

Increased Use of Mathematical Symbols

The same tendency which is making for a reduction of verbal forms is increasing the use of mathematical symbols. As logical relations are less frequently written out, a simple sign such as + or ÷ is used. The algebraic x is supplied in place of a whole roundabout series of awkward preliminary statements or assumptions. With it, of course, come changed methods of manipulation, as in the use of the algebraic equation.

It is doubtless true that the rigidity of full logical forms is giving way to a more flexible and natural mode of expressing the child's thoughts. Fixed oral and written forms of exposition may assist the child, much as the acquisition of a definite symbol fixes an abstract meaning, which remains unwieldy till it attaches itself to a word by which it is to be recalled. But increasing care

is manifested that children shall use only those forms that will conform to practical need upon the one hand, and to natural, efficient, and economical mastery on the other.

XI

EXAMPLES AND PROBLEMS

Formal and Applied Arithmetic

THE teaching of arithmetic is usually classified under two aspects, formal work and applied work. The formal work deals mainly with the memorization of fundamental facts, processes, and other details of manipulation. The applied work, as the name implies, is the formal work utilized in the setting of a concrete situation demanding a solution. These two aspects of arithmetical instruction are very frequently sharply separated, the child working alternately with one or the other. The characteristic practice is to deal with them without relating them as closely as the highest efficiency would demand.

The Example and the Problem

Formal exercises in arithmetic are usually presented through the "example"; the exercises in application through the "problem"; the distinc-

tion being that one is an abstract and symbolical statement of numerical facts and the other a concrete and descriptive statement.[1] In the first case the mathematical sign tells the child *what* to do, whether to add, subtract, multiply, or divide; the example being a kind of pre-reasoned problem, the child has only to manipulate according to the sign, his whole attention throughout being focused on the formal calculation. In the second case the child has two distinct functions; he must, from the description of the situation presented, decide, through the process of reasoning, *what* he is to do (add, subtract, divide, or multiply), and having rendered his judgment, he must proceed through the formal calculation.

The Traditional Precedence of Formal Work

As the problem involves two types of mental processes in a single exercise, and the example

[1] While this distinction is not general, it has sufficient currency to warrant its use here for the convenience of discussion. The expression "clothed problem" (from the German) is occasionally used to mean what is here designated as "problem," and "abstract problem" is used to mean what is here designated as "example."

but one, the usual procedure in arithmetic is to take up the formal side through examples first and, later on, the applied side through the use of problems. This means that the first emphasis is on formal and abstract work rather than on a treatment of natural, concrete situations, an emphasis not wholly sanctioned by modern psychology and the better teaching procedure of other subjects.

Objective and Narrative Presentation as a Reform Tendency

The reform tendency is found mainly in the primary grades where the beginnings of new processes are made through objective presentations of the problem. But the transition from objectified problems to formal work is not immediate. The children pass from objectified situations to "number stories," which are only descriptions or narratives of a situation. This is the interesting primary-school equivalent of that more businesslike language description found in the higher grades, the arithmetical problem. But it precedes formal work and succeeds it, — the

formal drill being a mere intermediate drill. Here concrete presentations and formal work are more closely related and more naturally ordered.

This reform tendency, which began in the primary school, is extending to the higher grades, where it is no longer rare to find the attack upon a process preceded by careful studies of the concrete circumstances in which the process is utilized. In the case of interest, several days might be utilized in studying the institution of banking in all its more important facts and relations. Such an approach not only provides motive for the formal and mechanical work, but gives a necessary logical basis in fact. Hence the understanding of practical business life makes accurate reasoning possible for the child when he is called upon to solve actual problems in application of the formal work.

The Over-Emphasis of Formal Work

It is perfectly natural under the general traditional practice of putting the first emphasis on mastery of formal work that the largest amount of attention should be given to the mechanical

and technical side of arithmetic, and that the concrete uses and applications should be slighted, and this is generally true of the practice of American teachers. Much more ingenuity has been used in the careful training of the child on the formal side than in teaching him to think out his problems. There is no such careful arranging and ordering of types in teaching a child to reason, as there is in teaching him to calculate.

The Need for More Systematic Teaching of Reasoning

Here and there a few thoroughly systematic attempts are made to carry the pupil through the simple types of one-step reasoning, to two-step and three-step problems with their possible variations. Just as the example isolates the difficulties of calculation, by letting the sign + or − stand for the logic of the situation, there is a tendency to give problems without requiring the calculations. This affords a means of isolating and treating the special difficulties of reasoning. The child is merely required to tell what he

would do, without doing it; the answer being checked by the gross facts. A little later, as a transition, he is permitted to give a rapid, rough approximation of what the answer is likely to be. With further command he tells what he would do and does it *accurately*. But such a program of teaching is still rare among teachers.

Existing Devices for Testing Reasoning

The care of the child's reasoning is largely restricted to testing his comprehension of the problem, (1) by having him restate the problem to be sure he understands it, or (2) by having him give a formal oral or written analysis of the way in which he solved the problem. The first requirement may not be thoroughgoing, as the child may give a verbal repetition of the problem without really knowing its meaning. The second is a formal analysis of the finished result and does not represent the genetic method of the child's thinking. Consequently its formulas do not, in any considerable degree, assist him in his actual struggle with the complex of facts.

Sources of Failure in the Solution of Problems

This lack of a systematic teaching of the technique of reasoning is manifest in the unreliability of children's thinking. When a child fails in a problem assigned from the text-book, the source of the error may be in one or more of three phases: (1) In failing to get the meaning of the language used to describe the details of the situation; (2) in failing to reason out *what* needs to be done to solve the situation; (3) in failing to make an accurate calculation. The first is a matter of language; the second, one of reasoning; the third, one of memorization. The elimination of errors, due to the first and third sources, leaves a considerable proportion to be accounted for by the second. Such informal investigations as have been made seem to show that the children who fail in reasoning do not make any real effort to penetrate into the essential relations of the situation. They depend on their association of processes with specific words of relation used in the description of the problem, an association determined of course by their

past experiences. As long as these familar "cue" words are used, they succeed. Let unfamiliar words or phrases be utilized in their stead or let the relation be implied, and, like as not, the children will fail to do the right thing. Practical school people are familiar with the fact that children solve the problems given in the language of their own teachers and fail when the problems are set by principals or superintendents, whose language is strange to them.

The Need of Varied Presentations of Problems

A varied use of materials in the objective presentation of problems, and a more constantly varied use of language in the descriptive presentation of problems would prevent the child from making such superficial and unthoughtful associations, and force him to think out connections between what is essential in a typical problem and the appropriate process of manipulating it. But such a deliberate application of modern psychology is far from being a conspicuous minority movement. The subject-matter of the problems given to children has,

however, improved greatly. Obsolete, puzzling, and unreal situations, which only hinder the child's attempt to reason, are less and less used in problem work.

Improvement in the Subject-Matter of Problems

Daily it becomes recognized with greater clearness that right reasoning depends upon a comprehension of the facts of the case, and the facts of the case in point must be within the experience of the child. This is the only way in which a problem can be real and concrete to him.

Real and Concrete Problems Taken from the Larger Social World

The recent effort on the part of text-book writers and teachers to make arithmetical problems "real" and "concrete" has not always recognized the above-mentioned psychological principle. The terms "real" and "concrete" have been interpreted in many ways besides in terms of the child's consciousness. With some, "real" has meant "material"; and the problems,

more particularly with primary children, have, in increasing degree, been presented by objects or words connoting very vivid images. Others have defined this quality in terms of actual existence or use in the larger social world. If these problems actually occur at the grocer's, the banker's, or the wholesaler's, it is said that they "are indeed concrete." And much effort has been expended in carrying these current problems into the classroom, in spite of the fact that they may be neither comprehensible nor interesting to the pupil.

Real and Concrete Problems Taken from the Child's Own Life

There is another social world, nearer home to the child, from which a more vital borrowing can be made. There is an opportunity to use the child's life in its quantitative aspects, to take his plays, games, and occupations, and introduce their situations into his mathematical teaching. As his world expands from year to year, he will be carried by degrees from personal and local situations to those of general interest. The

teacher can provide this progression without devitalizing the facts presented.

The Imaginative or Hypothetical Problem

There is another error into which both the socially-minded radical and the specialist in child study fall. In their eagerness to improve the arithmetical problem, they assume that problems taken from the larger social world or from the child's experience are necessarily superior to hypothetical, imaginative, or "made-up" problems. The psychological fact that needs to be forced upon the attention of the reformers is that, with proper artfulness, an imagined problem may be even more vital and real to the child than one taken from life — as a situation in a drama may be more appealing and real to a child than one on the street. This has some recognition, but not enough. Those who stand upon the side of the "made-up" problems are more likely to be reactionaries who tolerate the traditional type of problem even though its stupid artificiality is obvious to both the teacher and the child. They might better be dealing with dull problems bor-

rowed from real life than with specially invented dullness.

Valid Arguments for Actual Problems

Of course there is another argument for the use of actual social materials. The child must ultimately come into command of precisely these facts, since their mastery will be demanded by the business world. But must a primary school child study his arithmetic through problems taken from the dreary statistics of imports and exports merely because tariff reform is a political issue which every citizen ought finally to comprehend? There is a time for this, and, as is the case with most of such borrowed business problems, the time is later. In so far as these are current situations within the contacts of child life, let them enter. A quantitative revelation of life is important; and it is good teaching economy to gain knowledge by the way, provided it does not distract attention from whatever main business is at hand.

Unity in the Subject-Matter of Problems

The socializing of arithmetical problems has one other additional good effect. It has tended to bring some topical unity into the problems constituting the assignment for a given lesson, or group of lessons. Hitherto, a series of problems was almost always composed of a heterogeneous lot of situations. There was no unity save that some one process was involved in each. The movement is now in the direction of attaining a more approximate unity within the subject-matter of the problems themselves. The difficulties of attainment have restricted this movement to more progressive circles.

The Eclectic Source of Problems

The eclectic source of arithmetic problems is apparent from the foregoing discussion. It would seem that some better texts would naturally be evolved through the implied criticism of each movement upon the other. Such is the case. Problems from child life emphasize the beginning condition to which adjustment must be

made in all good teaching. Those from the greater world suggest the final goals of instruction. Those "made up" by the teacher call attention to what is too often forgotten, that the educative process in school may be artful without becoming artificial. Teaching is art, and when well done is not less effective for the fact.

XII

CHARACTERISTIC MODES OF PROGRESS IN TEACHING METHOD

Variation in Method, and its Causes

THE existing methods of teaching arithmetic in the American elementary schools are exceedingly varied. This is due to many causes. The democratic system of local control, as opposed to a centralized supervision of schools, has increased both the possibility and the probability of variation. Even within the units of supervision (state, county, and municipal) the opportunity for reducing variation in the direction of a more efficient uniformity is lost. This is partly due to the lack of a thoroughly trained staff of supervisors of the teaching process. Uniformity beyond the legal units of supervision has been restricted by the lack of organized professional means for investigation of and experimentation in controversial matters. Even such crude ex-

periments as are being tried in more than one class room, school, or system are unknown, unreported, unestimated, because no competent professional body gathers, evaluates, and diffuses such knowledge. In this respect the teaching profession is far below the efficient organization of the legal and medical professions.

Characteristic Traditions and Reforms

It is exceedingly difficult therefore to analyze the characteristic aspects of teaching method except as these are interpreted in movements of general significance. These may be actual or potential, traditional or reformatory, general or local in present acceptance. The situation is one wherein tradition is mixed with radicalism, and radicalism modified by reaction. In this medley of movements there are dominant tendencies both traditional and progressive.

Forces for Progress in Method

It is quite impossible to indicate the progressive tendencies with clearness save in connection with the discussions of concrete difficulties in

mathematical teaching. The forces that are behind these tendencies may, however, be summarized here. For convenience, they may be classified into eight types of influence, extending from more or less vague and general movements to very particular, scientific contributions. No attempt is made to indicate the achievement of each ; the form of each influence is only roughly defined, and illustrative movements or studies are suggested :—

General Pedagogical Movements

(1) It is obvious that any general pedagogical movement that influences the professional attitude of teachers will influence the special methods of mathematical teaching. The appearance of the doctrine of interest made mathematical instruction less formal. The growing enthusiasm for objective work enlarged the use of objects in the arithmetic period. The child study movement laid emphasis upon the child's own plays and games as a source of problems and examples.

Special Pedagogical Movements

(2) Certain special movements in methods of teaching, local to the subject of mathematics, have also been effective. Here one has only to recall the "Grube" method, with its influence on the order and thoroughness with which the elements of arithmetic are taught.

Daily Trial and Error

(3) The tendency of every teacher, who is at all sensitive to the defects of his methods, is to vary his daily practice. Constant trial, with error eliminating and success justifying a departure, is thus a source of progress. The new devices of one teacher are taken up by the eager professional witness, and innovation is thus diffused. We have no ability to measure how much professional progress is due to individual variation in teaching and its conscious and unconscious imitation. The disposition of school systems to send their teachers on tours of visitation without loss of salary is a recognition of the value of this method of advance.

Experimentation of Progressive Teachers

(4) A far more efficient and radical source of change than that just mentioned is the deliberate, conscious, experimental teaching of progressive individuals. Some new idea or device occurs to the teacher of original mind, and it is tried out with a fair proportion of resulting successes. An illustration of such a contribution is found in one conspicuous effort to get more rapid column addition. The first columns to be added were allowed to determine the selection and order of addition combinations learned. Thus if $6+7+9+6+7=35$ is the first column to be used, then the first combinations mastered will be $6+7=13$, $3+9=12$, $2+6=8$, $8+7=15$. Arising as a fruitful idea and seeming to give a measure of success, it has been carried, in the particular locality in mind, from school to school, and from system to system.

Reconstruction through Psychological Criticism

(5) A prolific source of radical change is found in the critical application of modern psychology

to teaching methods. Algorisms, types of difficulty, the order and gradation of these, as well as many other factors in method have been radically reorganized on psychological grounds. Examples of such psychological modifications of method are found in the "Courses of Study for the Day Elementary Schools of the City of San Francisco." Still more extensive critical applications are found in the "Exercises in Arithmetic" devised by Dr. E. L. Thorndike, Professor of Educational Psychology in Teachers College, Columbia University.

Studies in the Special Psychology of Mathematics

(6) Attempts have been made to inquire into the special psychology of arithmetical processes through careful experimentation and control. They have not been numerous, nor have they been influential on current practice. Such a field needs development. A typical attempt to investigate and formulate the special psychology of number is found in a Clark University study of

"Number and its Application Psychologically Considered."[1]

Investigations of Existing Methods

(7) Educational investigations as to the efficiency of existing arithmetical teaching among school systems, sufficiently varied to be representative of American practice, have also been conducted. These have usually gone beyond the field of the special methods of presentation employed in the classroom, and have inquired into the conditions of administration and supervision, the arrangement of the courses of study, and other similar factors. Dr. J. M. Rice's studies into "The Causes of Success and Failure in Arithmetic"[2] investigated such specific factors as: The environment from which children come, their age, time allotment of the subject, period of school day given to arithmetic, arrangement of home work, standards, examinations, etc. A subsequent

[1] Phillips, D. E., "Number and its Application Psychologically Considered," *Pedagogical Seminary*, 1897–8, vol. 5, pp. 221–281.

[2] Rice, J. M., "Educational Research: Causes of Success and Failure in Schools," *Forum*, 1902–03, vol. 34, pp. 281–97, 437–52.

study of similar type, but employing more refined methods, is that of Dr. C. W. Stone on "Arithmetical Abilities and some Factors determining them."[1] The main problem of this study was to find the correlation between types of arithmetical ability and different time expenditures and courses of study. These two studies have probably attracted more general notice than any other studies of arithmetical instruction. While they have largely dealt with administrative conditions that limit teaching method, rather than with the details of teaching method itself, they have stimulated the impulse to investigate conditions and practices of every type.

Special Experiments in Controlled Comparative Teaching

(8) The latest source of progress in teaching method is found in the movement for comparative experimental teaching under normal but carefully controlled conditions. Several such experiments are being conducted in the Horace Mann

[1] Stone, C. W., "Arithmetical Abilities and Some Factors determining them," *Columbia University Contributions to Education, Teachers College*, N. Y. City, 1909, p. 101.

Elementary School of Teachers College, Columbia University, under the direction of Principal Henry C. Pearson, with the co-operation of the staff of Teachers College. This experimental work is designed to determine primarily the relative value of competing methods in actual use throughout the country, the assumption being that every substantial difference in practice implies a difference of theory and consequently a controversy that can be resolved only on the basis of careful comparative tests. Two parallel series of classes of about the same age, ability, teacher equipment, etc., are selected for this work. One series is taught by one method; the other series by the other method. The abilities of these children are measured both before and after the teaching, and the growth compared. The standards and methods of this type of comparative experimentation, together with a list of current competitive methods requiring investigation, are given in Dr. David Eugene Smith's monograph on "The Teaching of Arithmetic."[1]

[1] Smith, D. E., "The Teaching of Arithmetic," chap. XVI, *Teachers College*, January, 1909.

OUTLINE

I. THE SCOPE OF THE STUDY

I. THE INFLUENCE OF AIMS ON TEACHING

OUTLINE

III. THE EFFECT OF THE CHANGING STATUS OF TEACHING METHOD

IV. METHOD AS AFFECTED BY THE DISTRIBUTION AND ARRANGEMENT OF ARITHMETICAL WORK

OUTLINE

THE DISTRIBUTION OF OBJECTIVE WORK

VI. THE MATERIALS OF OBJECTIVE TEACHING

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